6.1. Conjugate points. Let $c\colon [a,b] \to M$ be a geodesic such that for all $t \in (a,b]$ the point $c(t)$ is not conjugate to $c(a)$ along c. Show that for all $s,t \in [a,b]$ with $s < t$, we have that $c(t)$ is not conjugate to $c(s)$ along c.

Solution. Assume that there are $s < t$ such that $c(t)$ is conjugate to $c(s)$ along c. Then by Theorem 3.16 there is some proper variation $(\gamma_r)_{r\in(-\epsilon,\epsilon)}$ of $\gamma_0 := c|_{[s,t]}$ such that $L(\gamma_r) < L(\gamma_0)$ for $r \neq 0$.

Define now $c_r\colon [a,b] \to M$ with

$$c_r(t) := \begin{cases} \gamma_r(t), & t \in [s,t], \\ c(t), & t \notin [s,t]. \end{cases}$$

But then we have

$$\begin{aligned} L(c_r) &= L(c|_{[a,s]}) + L(\gamma_r) + L(c|_{[t,b]}) \\ &< L(c|_{[a,s]}) + L(c|_{[s,t]}) + L(c|_{[t,b]}) = L(c), \end{aligned}$$

contradicting Theorem 3.13. $\square$

6.2. Trace of a symmetric bilinear form. Let $(V, \langle\cdot,\cdot\rangle)$ be a m-dimensional Euclidean space and let $r\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form. Furthermore, let $S^{m-1} = \{v \in V : |v| = 1\}$ be the unit sphere. Prove that

$$\int_{S^{m-1}} r(v,v)\ d\mathrm{vol}^{S^{m-1}} = \frac{\mathrm{vol}(S^{m-1})}{m}\mathrm{tr}(r) = \omega_m \mathrm{tr}(r),$$

where $d\mathrm{vol}^{S^{m-1}}$ denotes the induced volume on S^{m-1} and ω_m is the volume of the m-dimensional unit ball.

Solution. Let $e_1,\dots,e_m$ be a orthonormal basis of V such that r is diagonal, that is $r(v,v) = \sum_{j=1}^m \lambda_j v_j^2$. Moreover, let $\tau_0,\dots,\tau_{m-1} \in \mathrm{SO}(m,\mathbb{R})$ be the isometries defined by $\tau_i e_j := e_{i+j \mod m}$. Then we have

$$\begin{aligned} \int_{S^{m-1}} r(v,v)\, d\mathrm{vol}^{S^{m-1}} &= \frac{1}{m}\sum_{i=0}^{m-1} \int_{S^{m-1}} r(\tau_i v, \tau_i v)\, d\mathrm{vol}^{S^{m-1}} \\ &= \frac{1}{m}\int_{S^{m-1}} \mathrm{tr}(r)\, d\mathrm{vol}^{S^{m-1}} = \frac{\mathrm{vol}(S^{m-1})}{m}\, \mathrm{tr}(r), \end{aligned}$$

since

$$\begin{aligned}\sum_{i=0}^{m-1} r(\tau_i v, \tau_i v) &= \sum_{i=0}^{m-1} r\left(\sum_{j=1}^{m} v_j e_{i+j}, \sum_{j=1}^{m} v_j e_{i+j}\right)\\ &= \sum_{j=1}^{m}\left(\sum_{i=0}^{m-1}\lambda_{i+j}\right) v_j^2\\ &= \operatorname{tr}(r)\sum_{j=1}^{m} v_j^2 = \operatorname{tr}(r).\end{aligned}$$

Finally, recall that $\omega_m = \int_0^1 r^{m-1}\operatorname{vol}(S^{m-1})\,dr = \frac{\operatorname{vol}(S^{m-1})}{m}$. □

6.3. Small balls and scalar curvature. Let p be a point in the m-dimensional Riemannian manifold (M, g). To goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r:

$$\operatorname{vol}(B_r(p)) = \omega_m r^m\left(1 - \frac{1}{6(m+2)}\operatorname{scal}(p) r^2 + \mathcal{O}(r^3)\right).$$

1. Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) := \exp_p(tv)$ and let $v, e_2, \dots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \dots m$. Show that the volume distortion factor of $\exp_p$ at tv is given by
$$J(v,t) := \sqrt{\det\left(\langle T_{tv}e_i, T_{tv}e_j\rangle\right)} = t^{-(m-1)}\sqrt{\det\left(\langle Y_i, Y_j\rangle\right)},$$
where $T_{tv} := (d\exp_p)_{tv}$.

2. Let $E_2, \dots, E_m$ be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is
$$Y_i(t) = tE_i - \sum_{k=2}^{m}\left(\tfrac{t^3}{6}R(e_i, v, e_k, v) + \mathcal{O}(t^4)\right)E_k.$$

3. Conclude that $J(v,t) = 1 - \frac{t^2}{6}\operatorname{ric}(v,v) + \mathcal{O}(t^4)$.

   Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon\operatorname{tr}(A) + \mathcal{O}(\epsilon^2)$.

4. Prove the above formula for $\operatorname{vol}(B_r(p))$.

Solution. 1. As we have seen in the proof of Proposition 3.6, the Jacobi fields Y_i are given as variation vector fields of $\alpha_i(s,t) := \exp_p(t(v+se_i))$, i.e.

$$Y_i(t) = \frac{d}{ds}\Big|_{s=0} \alpha_i(s,t) = T_{tv}te_i$$

and therefore $T_{tv}e_i = \frac{1}{t}Y_i(t)$.

Furthermore, we have $\langle T_{tv}v, T_{tv}e_i\rangle = \langle v, e_i\rangle = 0$ by the Gauss Lemma.

Then the volume distortion is given by

$$J(v,t) = \sqrt{\det\big(\langle T_{tv}e_i, T_{tv}e_j\rangle\big)} = t^{-(m-1)}\sqrt{\det\big(\langle Y_i, Y_j\rangle\big)}.$$

2. We check that the derivatives coincide. Clearly, we have $Y_i(0) = 0$, $\dot{Y}_i(0) = e_i$ and $\ddot{Y}_i(0) = -R(Y_i(0), \dot{c}(0))\dot{c}(0) = 0$. Furthermore,

$$\begin{aligned}\dddot{Y}_i(0) &= -(D_{\dot{c}}R)(Y_i(0), \dot{c}(0))\dot{c}(0) - R(\dot{Y}_i(0), \dot{c}(0))\dot{c}(0)\\ &= -R(e_i, v)v = -\sum_{k=2}^m \langle R(e_i,v)v, e_k\rangle e_k = -\sum_{k=2}^m R(e_k, v, e_i, v)e_k.\end{aligned}$$

3. With the above, we get

$$\begin{aligned}\langle Y_i, Y_j\rangle &= t^2\langle E_i, E_j\rangle - \tfrac{t^4}{6}\sum_{k=2}^m R(e_i, v, e_k, v)\,\langle E_k, E_j\rangle\\ &\qquad\qquad - \tfrac{t^4}{6}\sum_{k=2}^m R(e_j, v, e_k, v)\,\langle E_i, E_k\rangle + \mathcal{O}(t^5)\\ &= t^2\delta_{ij} - \tfrac{t^4}{3}R(e_i, v, e_j, v) + \mathcal{O}(t^5)\end{aligned}$$

and thus

$$\begin{aligned}J(v,t) &= \sqrt{\det\Big(\delta_{ij} - \tfrac{t^2}{3}R(e_i, v, e_j, v) + \mathcal{O}(t^3)\Big)}\\ &= \sqrt{1 - \tfrac{t^2}{3}\operatorname{tr}(R(e_i, v, e_j, v)) + \mathcal{O}(t^3)}\\ &= 1 - \tfrac{t^2}{6}\operatorname{ric}(v,v) + \mathcal{O}(t^3).\end{aligned}$$

4. First, we use polar coordinates:

$$\operatorname{vol}(B_r(p)) = \int_{B_r(0)} J(v,t)\,dx^1\dots dx^m = \int_0^r\int_{S^{m-1}} t^{m-1}J(v,t)\,d\mathrm{vol}^{S^{m-1}}\,dt.$$

Then, using exercise 2 and the above, we get

$$\begin{aligned}
\mathrm{vol}(B_r(p)) &= \int_0^r \int_{S^{m-1}} t^{m-1}(1 - \tfrac{t^2}{6}\,\mathrm{ric}(v,v) + \mathcal{O}(t^3))\,d\mathrm{vol}^{S^{m-1}}\;dt \\
&= \int_0^r t^{m-1}\left(\mathrm{vol}(S^{m-1}) - \tfrac{t^2}{6}\int_{S^{m-1}} \mathrm{ric}(v,v)\,d\mathrm{vol}^{S^{m-1}} + \mathcal{O}(t^3)\right)\,dt \\
&= \frac{r^m}{m} m\omega_m - \frac{r^{m+2}}{6(m+2)}\,\mathrm{scal}(p)\omega_m + \mathcal{O}(r^{m+3}) \\
&= \omega_m r^m \left(1 - \frac{r^2}{6(m+2)}\,\mathrm{scal}(p) + \mathcal{O}(r^3)\right).
\end{aligned}$$

$\square$